Regime of wave-packet self-action in a medium with normal dispersion of the group velocity

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Peculiar features of the self-action of non-one-dimensional wave packets described by the nonlinear Schrödinger equation with a hyperbolic spatial operator were studied analytically and numerically. It was shown that the self-action dynamics is determined by the consequence of the processes of transverse self-focusing filamentation and longitudinal splitting. Splitting scenarios were classified. It is shown that the strongest inhomogeneities are excited along ''hyperbolas'' in the self-similar collapse process.

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The nonlinear Shrödinger equation (NLSE) with a hyperbolic spatial operator [1]

$$i\frac{\partial u}{\partial t}+\Delta_{\perp}u-\frac{\partial^2 u}{\partial z^2}+|u|^2u=0 \tag{1}$$

describes a wide class of nonlinear wave processes. In the two-dimensional case (transverse-coordinate Laplacian $\Delta_{\perp}=\partial^2/\partial x^2$), it determines the self-action dynamics of a wave on a liquid surface [1–3] and the spatial evolution of wave packets in a magnetized plasma in the range of parameters that corresponds to a saddle surface of the refractive index [4–6]. However, recently of the greatest interest have been the studies of the ultrashort laser pulses self-action in the media with normal dispersion of the group velocity $\partial v_{\mathrm{gr}}/\partial\omega<0$ (in this case t corresponds to the coordinate of the wave packet centroid) by means of Eq. (1) [7–12]. The most attention was given to the fact that according to Eq. (1) the nonlinearity leads to the distortion of phase fronts, which causes, at the same time, wave focusing in the transverse direction and defocusing in the longitudinal one. The competition of these two processes can lead, in particular, to the stabilization of the transverse collapse of the axially symmetric wave beam existing in the case of $\partial/\partial z=0$ [7]. The possibility of arising of singularities during the evolution of the localized distribution is still the problem for discussion [7,10,13].

We consider another feature of the wave self-action in the framework of Eq. (1), which manifests itself just at the stage of modulation instability of the plane wave. The growth rate of such an instability is determined by the expression [4,5]

$$\gamma^2=(k_{\perp}^2-k_z^2)[2A_0^2-(k_{\perp}^2-k_z^2)], \tag{2}$$

where A_0 is the amplitude of the plane wave and $k_{\perp}$ and k_z are the longitudinal and transverse perturbation wave numbers. The value of γ reaches its maximum not on a sphere, as in the NLSE case, but on a hyperbolic surface

$$k_{\perp}^2-k_z^2=A_0^2, \tag{3}$$

i.e., in the framework of Eq. (1) instability is possible for any, however small, scales of $(L\simeq 1/k)$, so that the characteristic longitudinal $(L_{\parallel}\sim 1/k_z)$ and transverse $(L_{\perp}\sim 1/k_{\perp})$ modulation scales correspond to the relationship (3). Therefore, such an instability should be realized also either for modulation of a plane wave or even in the case of transverse self-focusing of the localized distribution and should cause splitting of the wave packet in the transverse and longitudinal directions.

Let us consider a new regime of self-action, which reflects peculiarities of the hyperbolic spatial operator, Eq. (1). By selecting initial distributions of the wave field specially, we will illustrate the possibility of the excitation of strong inhomogeneities during the splitting process, Eq. (1). Evidently, the regime is characterized by a strong spectrum broadening. When solving the problem numerically, we considered the plane $(\Delta_{\perp}\rightarrow\partial^2/\partial x^2)$ version of the initial equation (1). This helped, unlike in earlier papers [6–9,11] to observe the long-term evolution of the system. Analytical results have a wider applicability range, since, are relatively easily generalized for the three-dimensional case.

In the considered case of the hyperbolic two-dimensional spatial operator the self-similar field distribution takes place, which depends on $\zeta=x^2-z^2$. The equation that describes the evolution can be written in the form

$$i\frac{\partial u}{\partial t}+4\zeta\frac{\partial^2 u}{\partial\zeta^2}+4\frac{\partial u}{\partial\zeta}+|u|^2u=0. \tag{4}$$

At $\zeta>0$ the introduction of the new variable, $\eta=\sqrt{\zeta}$ transforms Eq. (4) to the equation, which is well known within the self-focusing theory

$$i\frac{\partial u}{\partial t}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\eta\frac{\partial u}{\partial\eta}+|u|^2u=0. \tag{5}$$

Under the corresponding conditions it describes the collapse in the ''focusing'' sector, $x^2-z^2=\zeta>0$. However, unlike the corresponding process in NLSE, the collapse direction is not toward the system axis $(x=0,z=0)$, but to some hyperbola, $x^2-z^2=\zeta_0>0$. It is evident that for the initial symmetric distribution there are two such hyperbolas and, hence, simultaneously with the collapse one can expect that the wave field will be stratified in the transverse direction, as in the case of the self-focusing instability in conventional NLSE.

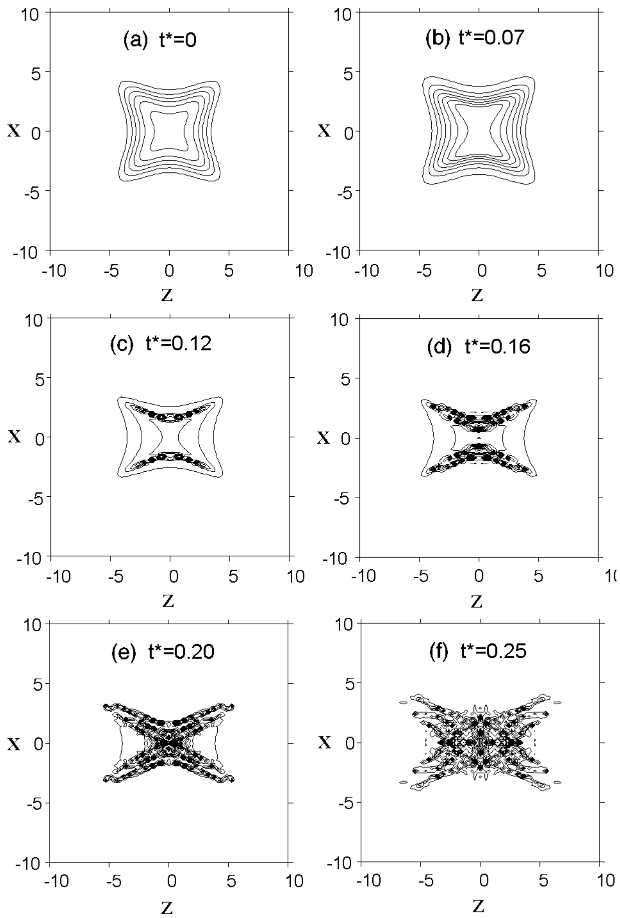

FIG. 1. Level lines of the wave field $|u(x,z,t=t^*)|^2$.

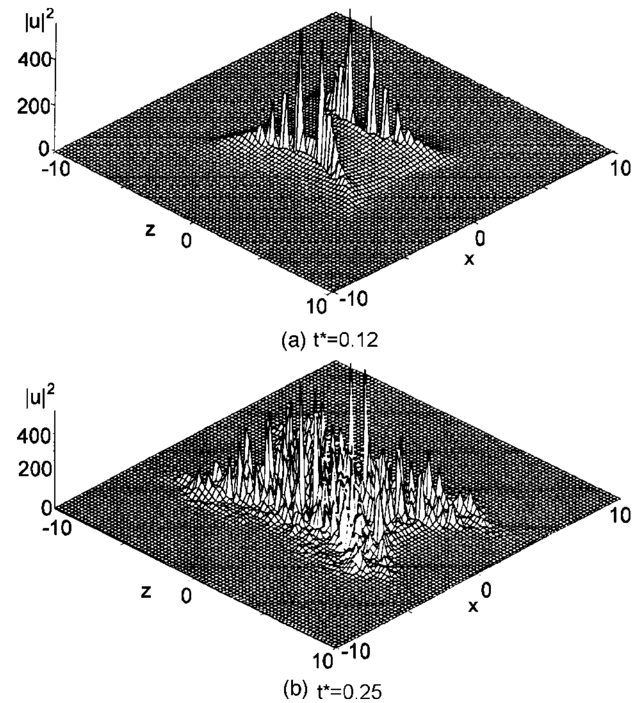

FIG. 2. Structure of the wave field $|u(x,z,t=t^*)|^2$. (a) $t^*=0.12$, (b) $t^*=0.25$.

In the sector $x^2-z^2=\zeta<0$ the similar transformation, $\eta=\sqrt{-\zeta}$ reduces Eq. (4) to Eq. (5) with the inverse sign of the nonlinearity ($|u|^2\rightarrow-|u|^2$), which describes the wave field self-defocusing. The different behavior of the solutions in the sectors $\zeta>0$ and $\zeta<0$ is characteristic for the equations with the wave (hyperbolic) operator.

One should specially note a peculiarity of Eq. (5) that determines the evolution of self-similar distributions. In the spatially two-dimensional case under consideration the initial equation, Eq. (1), describes the ''unidimensional'' self-focusing process weakened by a defocusing one. Contrariwise, Eq. (5) has the form of the ''two-dimensional'' NLSE, which proves that a ''strong'' self-focusing (collapse) is possible in the system under consideration. To analyze the decreasing of a collapse due to the defocusing it is useful to go over to the orthogonal coordinate system $\zeta=x^2-z^2;\mu=xz$. Thus, for these variables we have the following equation, equivalent to the initial one Eq. (1):

$$i\frac{\partial u}{\partial t}+4\left(\zeta\frac{\partial^2u}{\partial\zeta^2}+\frac{\partial u}{\partial\zeta}+2\mu\frac{\partial^2u}{\partial\zeta\,\partial\mu}-\zeta\frac{\partial^2u}{\partial\mu^2}\right)+|u|^2u=0. \tag{6}$$

Hence one can see that for the initial distribution with a weak dependence on μ, the derivatives over μ in the Eq. (6) can be omitted, so we come to Eq. (5) describing collapse. However, as the numerical calculations show, while collapsing the distribution becomes one-dimensional near the line $\zeta=\zeta_0$ and the role of the defocusing term $\zeta(\partial^2u/\partial\mu^2)$ is growing noticeable.

The most visible this effect is for the initial distributions with weak ''hyperbolicity''

$$u=\frac{u_0}{a}\exp\left\{-\frac{(x^2-z^2)^2}{2a^4}-\frac{x^2z^2}{2b^4}\right\}. \tag{7}$$

At $a\simeq b$ it is a wave field with its structure close to the distribution described by the product of the super-Gaussian functions ($\sim\exp[-(x^4+z^4/2a^4)]$). The level lines of the initial distribution (6) are presented in Fig. 1(a).

Figures 1–3 show the results of the computation of the initial distribution Eq. (7) evolution with $u_0=25,a=3,b=3.5$. The following characteristic stages are observed. First, rather slow and without noticeable changes in the maximum amplitude value the process of field transverse separation in two distributions [Figs. 1(a), 1(b)] is developed. As a result of this preliminary stage, the field becomes localized in the focusing sector [Fig. 1(b)]. Then the field sharply becomes stronger and localizes near the hyperbolas $x^2-z^2\simeq\pm3$. This process is accompanied by splitting of the field distribution near hyperbolas $\zeta\simeq\pm3$ in the defocusing direction [Fig. 1(c)]. Finally, rather strong field bunches [Fig. 2(a)] are formed.

In further evolution of the system one should note the process of some smoothing of longitudinal inhomogeneities and subsequent stratification in the transverse direction. As a result, the field becomes localized near two hyperbolas. The splitting instability evolution along these hyperbolas leads to formation of the field distribution shown in Fig. 1(d). This consequence of processes repeats several times [see Figs. 1(e),1(f)] the number of ''hyperbolas'' (six) corresponds to

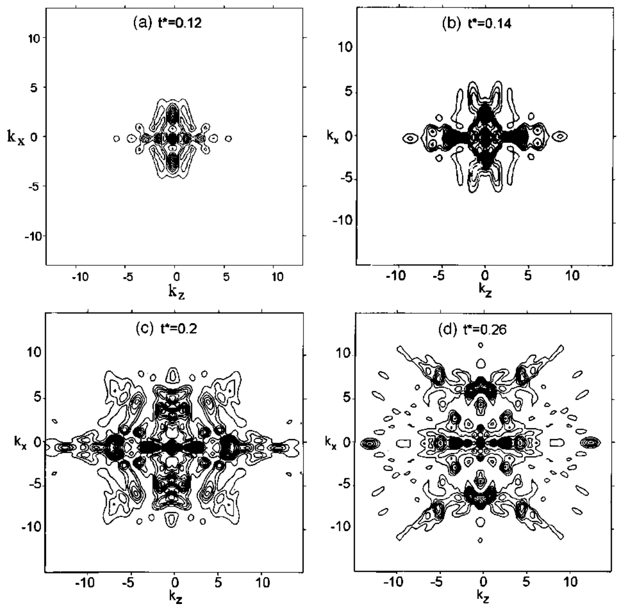

FIG. 3. Level lines of spatial spectrum of the wave field $|Fu(k_x,k_z,t=t^*)|^2$.

the development of filamentation instability [see Eq. (2) at $k_z=0$] and the near-axis region is filled with sharp sufficiently regular bunches [see Figs. 1(f),2(b)].

The picture of the numerical result level lines at the output moments is changed as a pattern in a kaleidoscope. The computations accuracy was controlled by the accuracy of integrals of Eq. (1):

$$I=\int |u|^2\, d\vec{r};H=\int \left(|\nabla_\perp u|^2-|u_z^2|^2-\frac{1}{2}|u|^4\right)d\vec{r}.$$

The study of further splitting evolution was stopped when the accuracy was reduced. 750×750 spatial Fourier harmonics of a spectral method were used to obtain the results. The calculating algorithm was realized on the Workstation with high capacity.

A peculiar feature of the dynamics of the self-action process under consideration is the anomalous broadening of the wave packet spectrum at the propagation. This effect can be interpreted as follows. In the process of self-focusing stratification the pattern becomes quasiunidimensional (see Fig. 1). In the framework of one-dimensional NLSE the action of the focusing nonlinearity leads to the downshift of the frequency of the distribution localized in the transverse direction (''soliton''), which is determined by the field amplitude. As a result of the splitting instability development [see Fig. 2(a)] the amplitude of the ''soliton'' becomes a periodic function along a hyperbola corresponding to the field maximum. This yields the phase modulation of the field in x,z coordinates and, hence, leads to a strong broadening of the spatial spectrum. The numerical calculations corraborate, that the development of the splitting instability is accompanied by a sharp broadening of the spatial spectrum. In addition, the spatial spectrum of the field amplitude is noticeably more narrow than the field spectrum, what is to the credit of the given interpretation.

The initial spectrum of the wave field $t=0$ is limited by the region of the central spot in Fig. 3(a). During the evolution the level lines of the spectrum look rather exotic (Fig. 3) due to nonlinear phase modulation of the field. At $t>0.22$ a spectrum excitation corresponding to the maximum of the instability increment of the plane wave of Eq. (3) is distinctively visible [Fig. 3(d)].

One should note one more peculiarity caused by the stratification of the field distribution in the focusing direction. The formation of a field minimum on the axis (see Fig. 1) due to this process leads to the formation of a concave phase front, which, evidently, retards the expanding of a wave field in the longitudinal direction. Thus, in spite of a common expanding tendency determined, for example, by the integral relationships [6–8], a defocusing process of the wave field along the z-axis turns to be noticeably stabled. Due to this fact one succeeds in keeping up the mean value of the field on the higher level in the central region where the instability develops.

Decreasing of u_0 in Eq. (7) leads to reducing of the number of hyperbolas in which vicinity the field is localized and at $u_0\leqslant 3$ ''hyperbolicity'' of the initial distribution ceases manifesting itself. The evolution of such a distribution ($u_0\leqslant 3$) proceeds as in Ref. [6].

Perturbations of a wave field localized near the characteristics of a hyperbolic operator of the original equation Eq. (1) can be excited for initial distributions without ''hyperbolicity,'' but in this case the process becomes slower. For example, in the case of a Gauss-shaped initial distribution, the longitudinal splitting is developed, first which results in the formation of two rather strong field maximums moving in the opposite directions along the z axis. As for the parameters corresponding to the considered distribution Eq. (7) the movement of the intensive localized field regions causes the excitation of shockwaves. The secondary splitting is developed along the front of these shockwaves so as near the hyperbolas (see Fig. 1).

We have considered a scenario for the dynamics of excitation of wave fields in the system described by the nonlinear Schrödinger equation Eq. (1) with the hyperbolic spatial operator. Three stages are characteristic for them: self-focusing filamentation, compression, and splitting of the filaments inhomogeneities in the defocusing direction.

The process of the wave field splitting in the longitudinal direction is accompanied by a strong spectrum broadening. The evolution of a wave field of the considered form (see Fig. 2) must be visible on a liquid surface.

It is readily seen that analytical results are easily generalized for the three-dimensional case. In this case the self-similar variable is $\zeta=r^2-z^2$. This gives grounds to be sure that the results obtained will not change in the 3D geometry. By that, one can expect that, especially in the self-similar regime of collapse to ''hyperboloids,'' the rate of splitting instability evolution will be noticeably higher and the generation of such sharp inhomogeneities makes it necessary to go beyond the framework of the ''quadratic dispersion'' approximation. The account of higher derivatives over z, evidently, playing the stabilizing role, requires additional study (see, for example, Ref. [14]). Thus, a hyperbolic character of the spatial operator in the nonlinear Schrödinger equation leads to the noticeable variety of the specific regimes of wave field self-action. In this paper we referred mainly to nonlinear optics of short laser pulses but the same picture has to occur in other fields described by Eq. (1), especially for the wave packets on the liquid surface [3].

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